Entropy and counting

Exercises

Exercise 1. Suppose that C is a binary code for $\mathcal{X}$, that is, an injective map $C\colon \mathcal{X} \to \{0,1\}^*$ from $\mathcal{X}$ to the set of $\{0,1\}$-sequences of finite length. Assume that the code is *uniquely decodable*, that is, the natural extension of C to $C^*\colon \mathcal{X}^* \to \{0,1\}^*$ via concatenation of strings is injective. Prove that every $\mathcal{X}$-valued random variable X satisfies

$$\mathbb{E}\left[\mathrm{length}\big(C(X)\big)\right] \geqslant H_2(X).$$

The index 2 above signifies that we measure entropy in bits (choosing $b = 2$ as the base of the logarithm).

Exercise 2. Prove that every discrete random variable X admits a prefix-free (and hence uniquely decodable) binary code of average length at most $H_2(X) + 1$.

Exercise 3. Prove the four basic inequalities and one basic equality involving entropy. Each inequality can be derived from the nonnegativity of the Kullback–Leibler divergence for appropriately defined pair of measures.

Exercise 4. A *Latin square of order* n is an $n \times n$ matrix $(L_{i,j})$ with entries from $[n]$ such that for every $i \in [n]$, the numbers $L_{i,1}, \ldots, L_{i,n}$ are all distinct and, similarly, for every $j \in [n]$, the numbers $L_{1,j}, \ldots, L_{n,j}$ are all distinct. Use Radhakrishnan's method to prove that asymptotically, as $n \to \infty$,

$$\#\text{Latin squares of order } n \leqslant \left(\frac{n}{e^2 - o(1)}\right)^{n^2}.$$

Exercise 5. Let K be a compact subset of $\mathbb{R}^n$. For each $j \in [n]$, let $\pi_j\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the projection map along the j^{th} coordinate. Prove that

$$\mathrm{vol}_n(K)^{n-1} \leqslant \prod_{i=1}^{n} \mathrm{vol}_{n-1}\big(\pi_j(K)\big).$$