Problem 1. Using the Lipton-Tarjan separator theorem, show an algorithm that checks if a given n-vertex planar graph G is hamiltonian in time $n^{\mathcal{O}(\sqrt{n})}$.

Problem 2. Recall that there is a noose version of the Lipton-Tarjan separator theorem: in a plane n-vertex graph there exists a noose γ that passes through $\mathcal{O}(\sqrt{n})$ vertices and leaves at most $2/3n$ vertices inside and outside the noose. Use this fact improve the running time from the previous problem to $2^{\mathcal{O}(\sqrt{n})}$.
Note: The crucial combinatorial observation you need to make is known in the literature under the name *Catalan structures*. This name is a hint to the exercise.

Problem 3. In the following problems, the input graph G comes with a tree decomposition of width t. For each problem, develop a dynamic programming algorithm that runs in time $f(t)n^{\mathcal{O}(1)}$ for as slowly growing function f as you can.

1. Dominating Set: find minimum size $X \subseteq V(G)$ such that $N[X] = V(G)$.
2. Steiner tree: given additionally $T \subseteq V(G)$, find a connected subgraph H of G with minimum number of edges that contains T.
3. Hamiltonian Cycle: find a connected subgraph of G where every vertex is of degree exactly 2.
4. Feedback Vertex Set: find minimum size $X \subseteq V(G)$ such that $G - X$ is a forest.
5. Planar Vertex Deletion: find minimum size $X \subseteq V(G)$ such that $G - X$ is planar.

Problem 4. In both problems below the input consists of a planar graph G and an integer k. Solve them in time $2^{\widetilde{\mathcal{O}}(\sqrt{k})}n^{\mathcal{O}(1)}$:

1. Feedback Vertex Set: does there exist $X \subseteq V(G)$ of size at most k such that $G - X$ is a forest?
2. Cycle Packing: do there exist k vertex-disjoint cycles in G?

Problem 5. Prove the Planar Grid Minor Theorem.

1. Consider a planar graph G with outer face surrounded by a simple cycle and with designated set T of $4k$ vertices on the outerface. Walk around the outerface in the clockwise direction, visiting all vertices of T and partition T into four sets of size k each; denote them N (north), E (east), S (south), and W (west). Assume that there exist k vertex-disjoint paths from N to S and k vertex-disjoint paths from W to E. Prove that G has a $k \times k$ grid as a minor.
2. Let G be a plane graph without $k \times k$ grid as a minor. Use the first point to recursively construct a tree decomposition of G of width $5k + \mathcal{O}(1)$. In a step of your recursion, you should maintain a plane graph G' with a set T' of at most $4k$ vertices on the outerface with a task to find a tree decomposition of G' with T' contained in one bag of the decomposition. If $|T'| = 4k$, then you can use a small separator between N and S or between W and E to make a recursive step.

Problem 6. Develop a polynomial-time algorithm for Max Cut in planar graphs. In this problem, given a graph G, we ask for a partition $V(G) = A \uplus B$ that maximizes the size of $E(A, B)$. An alternative — much more useful in this problem — way of thinking is to delete minimum number of edges from G to make it bipartite.

1. Consider the following T-Join problem: given a graph G and a set $T \subseteq V(G)$, one asks for a subgraph H of G with $V(H) = V(G)$ and of minimum possible number of edges such that every connected component of H has even number of vertices of T. Show that this problem is polynomial-time solvable by a reduction to a matching problem.
2. Reduce Max Cut in planar graphs to the T-Join problem in the dual graph.

Problem 7. Develop a fixed-parameter algorithm for Planar Vertex Deletion parameterized by the solution size. In this problem, we are given a graph G and an integer k and we ask if there exists a set $X \subseteq V(G)$ of size at most k such that $G - X$ is planar. We consider a seemingly simpler version, where we have access to a set $Y \subseteq V(G)$ of size exactly $k + 1$ such that $G - Y$ is planar.

1. Show that if the treewidth of $G - Y$ is t then one can solve Planar Vertex Deletion in time $f(t + k)n^{\mathcal{O}(1)}$ for some computable function f.
2. Show that if $G - Y$ has a $k^{100} \times k^{100}$ grid as a minor, then a vertex $y \in Y$ that is connected to many vertices of this grid that are far away from each other needs to be included in any sought solution X.
3. Show that in the absence of vertices from the previous point, $G - Y$ has a $k^{10} \times k^{10}$ grid as a minor such that only the outer layer of the grid can adjacent to other vertices of the grid and/or set Y.
4. Show that if one deletes a vertex in the middle branchset of the grid from the previous point then the answer to the problem does not chenge.
5. Deduce that the simpler version of the Planar Vertex Deletion problem (i.e., the one with the set Y) can be solved in time $2^{\mathcal{O}(k^c)}n^c$ for some universal constant c.
6. Deduce that the general version can be solved in similar time as well.